University of California, Santa Cruz
Department of Applied Mathematics and Statistics
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Classical and Bayesian Inference - AMS 132

Review and Example: Sampling distributions

1. Suppose that $X_1, ..., X_n$ form a random sample from the normal distribution with mean μ and variance σ^2, and Y is an independent random variable having the normal distribution with mean 0 and variance $4\sigma^2$.

   (a) Determine a function of $X_1, ..., X_n$ and Y that does not involve μ or σ^2 but has he t distribution with $n-1$ degrees of freedom.

   (b) If a sample of size 13 is considered, compute $P\left(\frac{Y}{2\sigma'} < 1.6\right)$, where $\sigma' = \sqrt{\frac{\sum_{i=1}^n (X_i - \overline{X}_n)^2}{n-1}}$.

   (c) If a sample of size 13 is considered, find values a and b, $a < b$ such that $P\left(a < \frac{Y}{2\sigma'} < b\right) = 0.94$.

2. Suppose that $X_1, ..., X_n$ are random variables describing the monthly income (in thousands of dollars) of people in the city of Santa Cruz. Assume that they form a random sample from the normal distribution with mean μ and variance σ^2.

   (a) Assume that both μ and σ^2 are unknown. Determine the smallest value of n such that the expected squared length of this interval will be less than $\sigma^2/2$.

   (b) If σ^2 is known, determine the smallest value of n such that the expected squared length of this interval will be less than $\sigma^2/2$.

   (c) Assume that both μ and σ^2 are unknown. From a sample of 25 person, the sample mean of their monthly income is 6.1 with a sample standard deviation of 1.2. Find a coefficient 0.9 confidence interval for the mean. The major has decided that a bonus will be given if the mean monthly income is less that 4 thousand dollars. What do you think the mayor will do based on this information?

   (d) Assume that σ^2 is known and equal to 1.5. Additionally, assume that a new person is considered in the sample. Find a coefficient 0.9 confidence interval for the monthly income of this new person.
   For this: find a pivot that has the standard normal distribution, and involves the monthly income of this new person and the other person in the sample. Then find random variables A and B such that $P(A < X_{n+1} < B) = 0.9$.

3. Suppose that $X_1, \ldots, X_7$ describe the age of 7 male students in a class and $Y_1, \ldots, Y_7$ describe the age of 7 female students in a class. In class, we assumed that $X_i \overset{iid}{\sim} N(\mu_1, \sigma^2)$, $Y_i \overset{iid}{\sim} N(\mu_2, \sigma^2)$, they are independent, and the variance is known an equal to 2. There is uncertainty whether the same

variance should be considered or not. For this, assume now that $X_1, \dots, X_7$ describe the age of 7 male students in a class and $Y_1, \dots, Y_7$ describe the age of 7 female students in a class, but now they are centered around 0 and have different unknown variance. So, $X_i \overset{iid}{\sim} N(0, \sigma_1^2)$, $Y_i \overset{iid}{\sim} N(0, \sigma_2^2)$, and they are independent. It is of interest to find a 90 percent confidence interval for the ratio of σ_2^2 and σ_1^2. For this:

(a) Show that $Z_1 = \sum_{i=1}^n \left(\frac{X_i}{\sigma_1}\right)^2 \sim \chi^2_{(n)}$, $Z_2 = \sum_{i=1}^n \left(\frac{Y_i}{\sigma_2}\right)^2 \sim \chi^2_{(n)}$, and they are independent. Give all the details on how you get this distributions.

(b) Find the random variable W such that $W\frac{\sigma_2^2}{\sigma_1^2}$ is a pivotal that has a Fisher distribution with n and n degrees of freedom.

Note: if X and Y are independent random variables such that $X \sim \chi^2_{m_1}$ and $Y \sim \chi^2_{m_2}$ then $\frac{X/m_1}{Y/m_2}$ follows a Fisher distribution with m_1 and m_2 degrees of freedom, denoted F_{m_1,m_2}.

(c) Consider $\gamma_1 = 0.05$ and $\gamma_2 = 0.95$, and compute $G^{-1}(\gamma_1)$ and $G^{-1}(\gamma_2)$, where G is the c.d.f. of a random variable that has a Fisher distribution with n and n degrees of freedom. You can use the table at the end of the book, page 862, or the R command, `qf(p, df1, df2` where `p` is the quantile, and `df1` and `df2` are the degrees of freedom.

(d) Find random variables A and B such that $P(A < \frac{\sigma_2^2}{\sigma_1^2} < B) = \gamma$.

(e) If $\sum_{i=1}^n x_i^2 = 5.46$ and $\sum_{i=1}^n y_i^2 = 11.05$, compute the γ percent confidence interval. What can you say about the equal variances assumption?